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Contents

Introduction

More algebraic structures were considered as a set of truth values, closed chains and complete lattices. In particular, the most general and most useful structure in current research is that of residual lattice.

The concept of residuated lattices was introduced in **1939** by **M. Ward** and **R. P. Dilworth** [19] as a generalization of the structure of the set of ideals of a ring. These algebras are a common structure among algebras associated with logical systems. The residuated lattices have interesting algebraic and logical properties.

A residuated lattice is an algebraic structure which consists both lattice and monoid structures, and has binary operations called residuations.

This work consist of three chapters.

In the first chapter, we present generalities on ordered sets, lattices and we finish with Boolean algebras.

In the second chapter, we study residuated lattices and some of its important algebraic proprieties.

In the last chapter, we study Heyting algebras and certain its proprieties.

Chapter 1

Generalities on ordered sets, lattices and boolean algebras

In this chapter, we present the most basic order-theoretic concepts as binary relations, ordered sets and lattices.

1.1 Binary relations and their properties

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements.

Definition 1.1 (Cartesian product).
Let E and F be two sets. The cartesian product of E and F written $E \times F$, is the set of pairs whose first element is from E and second is from F.

$$E \times F = \{(x, y) : x \in E \text{ and } y \in F\}.$$

Example 1.1.
Let $\{1, 3, 5\}$ and $\{2, 4, 6\}$ be two sets, then their cartesian product is
$\{1, 3, 5\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)\}$.

Definition 1.2 (Binary relation). *[13]*
Let E and F be two sets. A binary relation from E to F is a subset of $E \times F$.

Remark 1.1.

If $\Re$ is a relation from a set E to itself, that is, if $\Re$ is a subset of $E^2 = E \times E$, then we say that $\Re$ is a relation on E, $\Re$ called binary relation on E.

Example 1.2.

Let $E = \{1,2,3,4,5\}$. Define the following relation $\Re$ on E:

$$x\Re y \Longleftrightarrow x < y.$$

Then, $\Re = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$.

Definition 1.3 (Inverse relation). *[8]*

Let $\Re$ be any relation from a set E to a set F. The inverse of $\Re$ denoted by $\Re^{-1}$, is the relation from F to E which consists of those ordered pairs which, when reversed, belong to $\Re$ that is,

$$\Re^{-1} = \{(y,x) : (x,y) \in \Re\}.$$

Example 1.3.

let $E = \{1,2,3\}$ and $F = \{x,y,z\}$. Then the inverse of $\Re = \{(1,y),(1,z),(3,y)\}$ is $\Re^{-1} = \{(y,1),(z,1),(y,3)\}$.

Definition 1.4 (Composition of relation). *[13]*

Let E, F and G be sets, and let $\Re$ be a relation from E to F and let Σ be a relation from F to G. $\Re$ is a subset of $E \times F$ and Σ is a subset of $F \times G$. Then $\Re$ and Σ give rise to a relation from E to G denoted by $\Re \circ \Sigma$ and defined by:

$$a(\Re \circ \Sigma)c \text{ if for some } b \in F \text{ we have } a\Re b \text{ and } b\Sigma c.$$

That is,

$$\Re \circ \Sigma = \{(a,c)| \text{ there exists } b \in F \text{ for which } (a,b) \in \Re \text{ and } (b,c) \in \Sigma\}.$$

Definition 1.5 (The direct image and the reciprocal image).

Let $\Re$ be a binary relation on E, and X, Y two subset of E.

- *$\Re(X)$ is called the direct image of the X such that,*

$$\Re(X) = \{y \in E | \exists x \in X : x\Re y\}.$$

- $\Re^{-1}(Y)$ *is called the reciprocal image of the* Y *such that,*

$$\Re^{-1}(Y) = \{x \in E | \exists z \in Y : x\Re z\}.$$

Example 1.4.

Let $E = \{a, b, c, d\}$, $\Re = \{(a, a), (b, a), (c, b), (d, d)\}$, $X = \{a, c, d\}$ *and* $Y = \{a, b\}$, *then* $\Re(X) = \{a, b, d\}$ *and* $\Re^{-1}(Y) = \{a, c\}$.

Theorem 1.1. *[12]*

Let $\Re$ *be a relation from* E *to* F, E_1 *and* E_2 *two subsets of* E, *then*

(i) If $E_1 \subseteq E_2$, *then* $\Re(E_1) \subseteq \Re(E_2)$.

(ii) $\Re(E_1 \cup E_2) = \Re(E_1) \cup \Re(E_2)$.

(iii) $\Re(E_1 \cap E_2) \subseteq \Re(E_1) \cap \Re(E_2)$.

Proof :

(i) If $y \in \Re(E_1)$, then $x\Re y$ for some $x \in E_1$. Since $E_1 \subset E_2$, then $x \in E_2$, thus $y \in \Re(E_2)$. Then (i).

(ii) If $y \in \Re(E_1 \cup E_2)$, then by definition $x\Re y$ for some x in $E_1 \cup E_2$. If x is in E_1, then since $x\Re y$ we must have $y \in \Re(E_1)$. By the same argument, if x is in E_2, then $y \in \Re(E_2)$. In either case $y \in \Re(E_1) \cup \Re(E_2)$. Then $\Re(E_1 \cup E_2) \subseteq \Re(E_1) \cup \Re(E_2)$.
Conversely, since $E_1 \subseteq (E_1 \cup E_2)$, by (i) $\Re(E_1) \subseteq \Re(E_1 \cup E_2)$. Similarly, $\Re(E_2) \subseteq \Re(E_1 \cup E_2)$. Thus $\Re(E_1) \cup \Re(E_2) \subseteq \Re(E_1 \cup E_2)$. Then (ii).

(iii) If $y \in \Re(E_1 \cap E_2)$, then for some x in $E_1 \cap E_2$, $x\Re y$. Since x is in both E_1 and E_2, it follows that y is in both $\Re(E_1)$ and $\Re(E_2)$, that is $y \in \Re(E_1) \cap \Re(E_2)$. Then we have (iii).

Remark 1.2.

In general, the inverse inclusion in (iii) does not hold.

Example 1.5.

Let $E_1 = \{2\}$, $E_2 = \{1, 3\}$ *and* $\Re = \{(1, z), (2, y), (3, y)\}$. *We have* $\Re(E_1 \cap E_2) = \Re(\emptyset) = \emptyset$, $\Re(E_1) = \{y\}$ *and* $\Re(E_2) = \{y, z\}$, *then* $\Re(E_1) \cap \Re(E_2) = \{y\}$, *so* $\Re(E_1) \cap \Re(E_2) \nsubseteq \Re(E_1 \cap E_2)$.

Proposition 1.1.

Let $\Re$ and Σ two binary relations on E, then

(i) $\Re \subset \Sigma \Longrightarrow \Re^{-1} \subset \Sigma^{-1}$.

(ii) $(\Sigma \circ \Re)^{-1} = \Re^{-1} \circ \Sigma^{-1}$.

Proof :

(i) If $(y, x) \in \Re^{-1} \Longrightarrow (x, y) \in \Re \Longrightarrow (x, y) \in \Sigma \Longrightarrow (y, x) \in \Sigma^{-1}$.

(ii)

$$\begin{aligned}
(x, y) \in (\Sigma \circ R)^{-1} &\Longleftrightarrow (y, x) \in \Sigma \circ \Re \\
&\Longleftrightarrow \exists t \in E : y\Sigma t \text{ and } t\Re x \\
&\Longleftrightarrow \exists t \in E : t\Sigma^{-1}y \text{ and } x\Re^{-1}t \\
&\Longleftrightarrow \exists t \in E : x\Re^{-1}t \text{ and } t\Sigma^{-1}y \\
&\Longleftrightarrow (x, y) \in (\Re^{-1} \circ \Sigma^{-1}).
\end{aligned}$$

Definition 1.6.

Let $\Re$ a binary relation on E, then $\Re$ is:

(1) reflexive if $(x, x) \in \Re$, *for all* $x \in E$.

(2) irreflexive if $(x, x) \notin \Re$, *for all* $x \in E$.

(3) symmetric if $(x, y) \in \Re \Longleftrightarrow (y, x) \in \Re$, *for all* $x, y \in E$.

(4) antisymmetric if $(x, y) \in \Re$ *and* $(y, x) \in \Re \Longrightarrow x = y$, *for all* $x, y \in E$.

(5) asymmetric if $(x, y) \in \Re \Longrightarrow (y, x) \notin \Re$, *for all* $x, y \in E$.

(6) transitive if $(x, y) \in \Re$ *and* $(y, z) \in \Re \Longrightarrow (x, z) \in \Re$, *for all* $x, y, z \in E$.

1.2 Ordered sets

1.2.1 Order and strict order

Definition 1.7 (Order).

A partial order (or just an order) on a non-empty set E is a binary relation $\leq$ on E that is *reflexive, antisymmetric and transitive. In this case, the pair $(E, \leq)$ is said to be a partially ordered set or poset (an ordered set).*

Example 1.6.

1. *Let A be a collection of subsets of set S. The relation $\subseteq$ is a partial order on A, so $(A, \subseteq)$ is a poset.*

2. *Let $\mathbb{Z}^+$ be the set of positive integers. The usual relation $\leq$ (less than or equal to) is a partial order on $\mathbb{Z}^+$, so $(\mathbb{Z}^+, \leq)$ is a post.*

Definition 1.8 (**Strict order**).
A relation $<$ on E is a strict order if it is irreflexive and transitive.

Definition 1.9 (**Total order**).
A relation $\Re$ on E is a total order if every $x, y \in E$ are comparable, that is, $x \leq y$ or $y \leq x$. In this case, the pair $(E, \leq)$ is said a totally ordered.

Definition 1.10 (**Dual order**).
Let $(E, \leq)$ an ordered set, we can define a relation denoted by $\geq$, such that

$$\forall x, y \in E, (x, y) \in \geq \Longleftrightarrow (y, x) \in \leq .$$

Definition 1.11 (**Chain and antichain**). *[18]*
Let $(E, \leq)$ be a poset.

- *A non-empty subset S of E is a chain in E if S is totally ordered by $\leq$.*

- *A non-empty subset S of E is an antichain in E if every two elements of S are incomparable.*

Example 1.7.

1. *$\mathbb{N}$ with the usual order is a chain.*

2. *$\mathcal{P}(E)$ with the inclusion relation does not a chain.*

1.2.2 Morphism of ordered sets

Definition 1.12 (**Morphism**)**.** *[19]*
If $(E, \leq_E)$ and $(F, \leq_F)$ are two ordered sets, then we say that a mapping f: $E \longrightarrow F$ is a morphism (isotone or order-preserving) if

$$(\forall x, y \in E) \ x \leq_E y \Longrightarrow f(x) \leq_F f(y).$$

An antitone (or order-inverting) if

$$(\forall x, y \in E) \ x \leq_E y \Longrightarrow f(x) \geq_F f(y).$$

Example 1.8.
Let f: $(D(6), |) \longrightarrow (D(30), |)$ a map define by $f(n) = 2n$, $\forall n \in D(6)$. f is a morphism.

Definition 1.13 (**Isomorphism**)**.**
Let $(E, \leq_E)$ and $(F, \leq_F)$ be two ordered sets, a mapping f: $E \longrightarrow F$ is called an isomorphism if

1. *$\forall x, y \in E$, $x \leq_E y \Longleftrightarrow f(x) \leq_F f(y)$.*
2. *f is bijective.*

1.2.3 Extremal elements of ordered sets

Definition 1.14 (**Maximal and minimal elements**)**.** *[18]*
Let $(E, \leq)$ be a partially ordered set.

- *A maximal element is an element $m \in E$ that is not less than in any other element of E that is,*

$$x \in E, m \leq x \Longrightarrow m = x.$$

  A maximum (largest or greatest) element m in E is an element that greater than in any other element of E that is,

$$x \in E \Longrightarrow x \leq m.$$

- *A minimal element is an element $n \in E$ that does not greater than in any other element of E that is,*

$$x \in E, x \leq n \Longrightarrow x = n.$$

A minimum (smallest or least) element n in E is an element less than in all other elements of E that is,

$$x \in E \Longrightarrow n \leq x.$$

Example 1.9.

1. The poset $\mathbb{Z}$ with the usual partial order $\leq$ has no maximal elements and has no minimal elements.

2. The poset $\mathbb{R}^+$ with the usual partial order $\leq$ has 0 a minimal element and has no maximal elements.

Definition 1.15 (**Upper and lower bounds**)**.** *[12]*

Consider a poset E and a subset A of E.

- *An element $x \in E$ is called upper bound of A if $a \leq x$ for all $a \in A$.*
- *An element $x \in E$ is called lower bound of A if $x \leq a$ for all $a \in A$.*

Definition 1.16 (**Supremum and infimum**)**.** *[12]*

Let E be a poset and A a subset of E.

- *An element $x \in E$ is called a least upper bound (supremum) of A if x is an upper bound of A and $x \leq a$, whenever a is an upper bound of A.*
- *An element $x \in E$ is called a greatest lower bound (infimum) of A if x is a lower bound of A and $a \leq x$, whenever a is a lower bound of A.*

1.2.4 Hasse diagram

Definition 1.17. *[18]*

Let $(E, \leq)$ be a poset. Then y covers x in E if $x < y$ and there is no element in lies strictly between x and y, that is

$$x \leq z \leq y \Longrightarrow z = x \text{ or } z = y.$$

Definition 1.18 (**Hasse diagram**)**.**

Many ordered sets can be represented by means of a Hasse diagram. In such a diagram we represent elements by points, we join the points representing x and y by an increasing line segment.

Example 1.10.

Let $E = \{1, 2, 3, 4, 6, 12\}$ *be the set of positive divisors of* 12*. If we order* E *in the usual way, we obtain a chain. If we order* E *by divisibility, we obtain the Hasse diagram:*

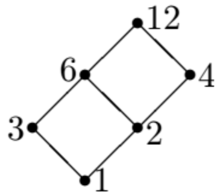


Figure 1.1

1.3 Lattices

Many important properties of an ordered set E are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of E. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices.

1.3.1 Basic notions

We shall be particularly interested in ordered sets $(X, \leq)$ in which $sup\{x, y\}$, $inf\{x, y\}$ exist for all $x, y \in X$.

Notation 1.1.

Looking ahead, we shall adopt the following neater notation: we write $x \vee y$ *in place of* $sup\{x, y\}$ *when it exists and* $x \wedge y$ *in place of* $inf\{x, y\}$ *when it exists. Similarly, we write* $\vee S$ *(the join of* S*) and* $\wedge S$ *(the meet of* S*) instead of* $supS$ *and* $infS$ *when these exist.*

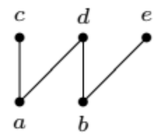
Definition 1.19. *[7]*

Let $(X, \leq)$ *be an ordered set.*

- *If* $x \vee y$*,* $x \wedge y$ *exist for all* $x, y \in X$*, then* $(X, \leq)$ *is called a lattice.*
- *If* $\vee S$*,* $\wedge S$ *exist for all* $S \subseteq X$*, then* $(X, \leq)$ *is called a complete lattice.*

Example 1.11.

1. $(\mathbb{N}^*, |)$ *is a lattice with* $x \vee y = ppcm(x, y)$ *and* $x \wedge y = pgcd(x, y)$ *for all* $x, y \in \mathbb{N}^*$.
2. *Every chain is a lattice such that* $x \vee y = max(x, y)$ *and* $x \wedge y = min(x, y)$.
3. *Let* E *in the Hasse diagram follow:*



E

Figure 1.2

E *is not a lattice.*

Definition 1.20.

Let L *be a lattice and* $A \subseteq L$. A *is called sublattice of* L *if for all* $x, y \in A$, $x \wedge y \in A$ *and* $x \vee y \in A$.

Theorem 1.2. *[7]*

Let L *be a lattice. Then for all* $a, b, c \in L$, $\vee$ *and* $\wedge$ *satisfy,*

$$\text{(L1)} \begin{cases} a \vee (b \vee c) = (a \vee b) \vee c \\ a \wedge (b \wedge c) = (a \wedge b) \wedge c \end{cases} \qquad \text{(associative laws)}$$

$$\text{(L2)} \begin{cases} a \vee b = b \vee a \\ a \wedge b = b \wedge a \end{cases} \qquad \text{(commutative laws)}$$

$$\text{(L3)} \begin{cases} a \vee a = a \\ a \wedge a = a \end{cases} \qquad \text{(idempotency laws)}$$

$$\text{(L4)} \begin{cases} a \vee (a \wedge b) = a \\ a \wedge (a \vee b) = a \end{cases} \qquad \text{(absorption laws)}$$

Proof :

(L1)

• We prove $a \vee (b \vee c) = (a \vee b) \vee c$:

Let $s = a \vee (b \vee c)$, then we have $a \leq s$ and $b \vee c \leq s$, then $b \leq s$ and $c \leq s$, hence $a \vee b \leq s$ and $c \leq s$. So s is an upper bound of the $\{a \vee b, c\}$.

Let M be an upper bound of the $\{a \vee b, c\}$ such that $M \neq s$, then $a \vee b \leq M$, hence $a \leq M$ and $b \leq M$, and we have $c \leq M$, then $a \leq M$ and $b \vee c \leq M$, then $s = a \vee (b \vee c) \leq M$, hence s is a least upper bound of the $\{a \vee b, c\}$. So $a \vee (b \vee c) = (a \vee b) \vee c$.

• The same method for $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(L2) and (L3) are evident.

(L4)

• We prove $a \vee (a \wedge b) = a$:

We have $a \wedge b \leq a$ and $a \leq a$, then a is an upper bound of the $\{a, a \wedge b\}$.

Let M be an upper bound of the $\{a, a \wedge b\}$ such that $M \neq a$, then $a < M$ and $a \wedge b \leq M$, hence a is a least upper bound of the $\{a, a \wedge b\}$. So $a \vee (a \wedge b) = a$.

• The same method for $a \wedge (a \vee b) = a$.

Theorem 1.3. *[7]*

*Let $(L, \wedge, \vee)$ be a non-empty set equipped with two binary operations which satisfy (L1) and (L2) from **Theorem** 1.2.*

(i) For all $a, b \in L$, we have $a \vee b = b$ if and only if, $a \wedge b = a$.

(ii) Define $\leq$ on L by $a \leq b$ if $a \vee b = b$. Then $\leq$ is an order relation.

(iii) With $\leq$ as in (ii), $(L, \leq)$ is a lattice in which the original operations agree with the induced operations that is, for all $a, b \in L$,

$$a \vee b = sup\{a, b\} \text{ and } a \wedge b = inf\{a, b\}.$$

Proof

(i) Assume $a \vee b = b$. Then,

$a = a \wedge (a \vee b)$ (by (L4))

$a = a \wedge b$ (by assumption)

Conversely, assume $a \wedge b = a$. Then

$b = b \vee (b \wedge a)$ (by (L4))

$= b \vee (a \wedge b)$ (by (L2))

$= b \vee a$ (by assumption)

$= a \vee b$ (by (L2))

(ii) Now define $\leq$ as in (ii). Then $\leq$ is reflexive by $(L3)$, antisymmetric by $(L2)$ and transitive by $(L1)$.

(iii) Let $sup\{a, b\} = s = a \vee b$, we have $a \leq s$ and $b \leq s$, then $a \vee b \leq s$. So s is an upper bound of the $\{a, b\}$.

Let M be an upper bound of the $\{a, b\}$ and $M \neq s$, then $a \leq M$ and $b \leq M$, hence $a \vee b = s \leq M$, then s is a least upper bound of the set $\{a, b\}$. So $sup\{a, b\} = s = a \vee b$.

The characterization of inf is obtained by duality (again).

1.3.2 Morphism of lattices

Definition 1.21 (**Morphism**). *[7]*

Let L_1 and L_2 be lattices. A map f: $L_1 \longrightarrow L_2$ is said to be a morphism if f is join-preserving and meet-preserving, that is, for all $a, b \in L_1$

$$f(a \vee b) = f(a) \vee f(b) \textbf{ and } f(a \wedge b) = f(a) \wedge f(b).$$

Proposition 1.2. *[18]*

A monotone map f: $L_1 \longrightarrow L_2$ between lattices need not in general preserve meets and joins. Even an order embedding need not preserve meets and joins.

Example 1.12.

Let the lattices L_1 and L_2 of integers defined by the Hasse diagram follow:

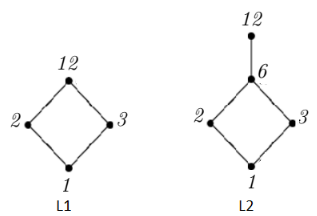


Figure 1.3

The map f: $L_1 \longrightarrow L_2$ defined by $f(n) = n$ is an order embedding, but

$$f(2 \vee 3) = f(12) = 12.$$

and

$$f(2) \vee f(3) = 2 \vee 3 = 6.$$

Hence, f does not preserve joins.

Definition 1.22 (**Isomorphism**).

Let L_1 and L_2 be two lattices and f: $L_1 \longrightarrow L_2$. f is called an isomorphism of lattices if,

1. *f is a morphism of a lattice.*
2. *f is bijective.*

1.3.3 Filters and ideals

Particularly important sublattices of a lattice are filters and ideals.

Definition 1.23 (**Filter**). *[7]*

Let L be a lattice. A non-empty subset F of L is called a filter if,

(F1) $a, b \in F$ implies $a \wedge b \in F$;

(F2) $a \in L, b \in F$ and $a \geq b$ imply $a \in F$.

Remark 1.3.

- *A filter is called proper if it does not coincide with L, or called improper.*

Definition 1.24 (**Principal filter**). *[18]*

Let L be a lattice and $F \subset L$, F is called principal filter if there is $a \in L$ such that $F = F_a$ and

$$F_a = \{x \in L : x \geq a\}.$$

Definition 1.25 (**Filter generated by a subset**). *[18]*

Let L a lattice and $G \subset L$ such that $G = \{a_1, ..., a_n\}$. The filter generated by G denoted by F_G is the smallest filter of L containing G, that is,

$$F_G = \{x \in L / \exists a_1, ..., a_n \in G, n \in \mathbb{N}^* : x \geq a_1 \wedge ... \wedge a_n\}.$$

Remark 1.4.

- *If* $G = \{a\}$*, then* $F_G = \{x \in L : x \geq a\} = F_a$*.*
- *If* L *is a finite lattice, then any filter is principal such that,* $F_G = F_{\wedge G}$*.*
- G *is called* $\wedge$*-incompatible if* F_G *is improper, i.e. there is a finite number of* G*,* $a_1, ..., a_n$ *such that* $a_1 \wedge ... \wedge a_n = 0$*.*
- G *is called* $\wedge$*-compatible if* F_G *is a proper filter.*

Definition 1.26. *[18]*

- *A proper filter* F *is maximal if for any filter* $X \subset L$*,*

$$F \subseteq X \subseteq L \Longrightarrow X = F \text{ or } X = L.$$

  A maximal filter is also called an ultrafilter.

- *A proper filter* F *is prime if*

$$a \vee b \in F \Longrightarrow a \in F \text{ or } b \in F.$$

Proposition 1.3. *[17]*
Let F *a proper filter, the following assertions are equipped:*

1. F *is an ultrafiltre.*
2. *For all* $x \notin F$*, there is* $y \in F$ *such that* $x \wedge y = 0$*.*

Definition 1.27 (Ideal). *[7]*
Let be a lattice. A non-empty subset I *of* L *is called an ideal if*

(I1) $a, b \in I$ *implies* $a \vee b \in I$*;*

(I2) $a \in L, b \in I$ *and* $a \leq b$ *imply* $a \in I$*.*

Remark 1.5.
Let L *be a lattice, then*

- *An ideal* I *is called proper if* $I \neq L$ *(i.e.* $1 \notin I$*).*

- *If* $a \in L$, $I_a = \{x \in L : x \leq a\}$ *is an ideal called principal ideal generated by* a.
- $G \subseteq L$, *the intersection* I_G *of all ideals containing* G *is an ideal namely ideal generated by* G.
- G *is called* $\vee$*-incompatible if* I_G *is an improper ideal, i.e. there is a finite number of* G, $a_1, ..., a_n$ *such that* $a_1 \vee ... \vee a_n = 1$.
- G *is called* $\vee$*-compatible if* I_G *is a proper ideal.*

Definition 1.28. *[18]*
Let L *be a lattice, then*

- *A proper ideal* I *of* L *is maximal if for any ideal* J,

$$I \subseteq J \subseteq L \Longrightarrow J = I \text{ or } J = L.$$

- *A proper ideal* I *is prime if,*

$$a \wedge b \in I \Longrightarrow a \in I \text{ or } b \in I.$$

Proposition 1.4. *[17]*
Let I *a proper ideal, the following assertions are equipped:*

1. I *is an ideal maximal.*
2. *For all* $x \notin F$, *there is* $y \in I$ *such that* $x \vee y = 1$.

1.3.4 Main algebraic properties of lattice

Definition 1.29 (**Closed lattice**).
A lattice is called closed if it has a least element denoted 0 *and greatest element denoted* 1.

Definition 1.30 (**Distributive lattice**). *[18]*
A lattice L *is distributive if it satisfies the distributive laws: for all* $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Example 1.13.

1. *Every sublattice of a distributive lattice is distributive.*
2. *Any chain is a distributive lattice.*
3. *The lattice* $(\mathbb{N}^*, |)$ *is distributive.*
4. $(\mathcal{P}(E), \cap, \cup, \subseteq)$ *is a distributive lattice.*
5. *The lattices* N_5 *and* M_3 *are not distributive.*

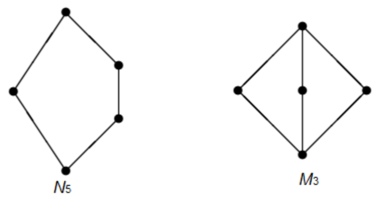


Figure 1.4

Remark 1.6. *[1]*

- *the only distributive lattices with the five elements are:*

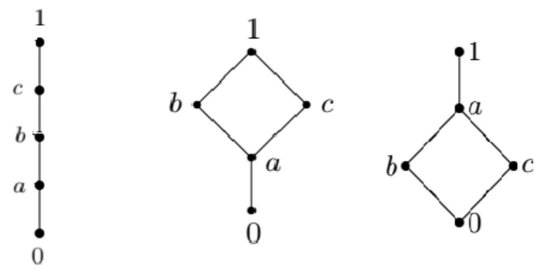


Figure 1.5

Theorem 1.4. *[18]*

If either of the distributive laws holds for all elements of a lattice L, then so does the other.

Proof :

Suppose that the first distributive law holds. Then applying it to the right side of the second distributive law and using absorption gives,

$$
\begin{aligned}
(a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \\
&= a \vee [(a \vee b) \wedge c] \\
&= a \vee [(a \wedge c) \vee (b \wedge c)] \\
&= a \vee (b \wedge c).
\end{aligned}
$$

Which shows that the second law holds.

Theorem 1.5. *[19]*

A lattice L is distributive if and only if, it has no sublattice of either of the forms M_3 and N_5.

Example 1.14.

*Let L be a lattice defined by the Hasse diagram in **Figure 1.6***

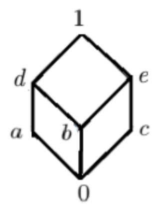


Figure 1.6

L is not distributive lattice because it contains a sublattice T of the form N_5 ($T = \{0, a, c, e, 1\}$).

Definition 1.31 (Modular lattice). *[18]*

A lattice L is modular if it satisfies the modular law: For all $a, b, c \in L$,

$$a \geq c \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Example 1.15.

1. Any distributive lattice is a modular lattice.

2. Every sublattice M of a modular lattice L is modular.

3. The lattice N_5 is modular but M_3 is not modular.

Theorem 1.6. *[19]*

A lattice L is modular if and only if, it has no sublattice of the form M_3.

Example 1.16.

*Let L be a lattice defined by the Hasse diagram in **Figure 1.7**:*

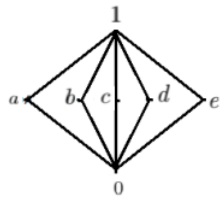


Figure 1.7

L is not modular lattice because it contains a sublattice S of the form M_3 ($S = \{0, b, c, d, 1\}$).

Theorem 1.7. *[17]*

A lattice L is modular if and only if, $\forall x, y, z \in L$

$$\begin{cases} x \wedge z = y \wedge z \\ x \vee z = y \vee z \end{cases} \Longrightarrow x = y \text{ or } x \parallel y.$$

Theorem 1.8. *[17]*

L is a distributive lattice if and only if, $\forall x, y, z \in L$

$$\begin{cases} x \wedge z = y \wedge z \\ x \vee z = y \vee z \end{cases} \Longrightarrow x = y.$$

Definition 1.32 (**Complemented elements**)**.** *[19]*

Let L is a bounded lattice, we say that $y \in L$ is a complement of $x \in L$ if $x \wedge y = 0$ and $x \vee y = 1$. In this case, we say that x is a complemented element of L.

Definition 1.33 (**Complemented lattice**)**.** *[19]*

A lattice L is complemented if every element of L is complemented.

1.4 Boolean algebras

Just as a lattice can be defined algebraically as a non-empty set with two operations $\wedge$ and $\vee$ satisfying certain requirements, with no explicit mention of the underlying partial order.

Definition 1.34 (Boolean algebra). *[18]*

A Boolean algebra is a complemented distributive lattice.

Proposition 1.5.

In a distributive lattice, the complement is unique if there exist. That is if x' and x'' are complements of the same element x then $x' = x''$.

Proof :

Suppose for $x \in L$, we have two complement x' and x'' ,then

$$\begin{cases} x \wedge x' = 0 \\ x \vee x' = 1 \end{cases} \text{and} \begin{cases} x \wedge x'' = 0 \\ x \vee x'' = 1 \end{cases}$$

$$\text{Then} \begin{cases} x \wedge x' = x \wedge x'' \\ x \vee x' = x \vee x'' \end{cases} \Longrightarrow x' = x''.$$

Example 1.17.

1. $(\mathcal{P}(X), \subseteq)$ *is a Boolean algebra.*

2. A bounded chain T is a Boolean algebra if and only if, $|T| \leq 2$.

3. $(\mathbb{N}^*, |)$ *is not a Boolean algebra.*

Theorem 1.9. *[18]*

Let L be a Boolean lattice and $a, b \in L$, then

(i) $0' = 1, 1' = 0$ *and* $a'' = a$.

(ii) (De Morgan's laws)

$$(a \wedge b)' = a' \vee b'$$

$$(a \vee b)' = a' \wedge b'$$

(iii) (Order and complements)

$$a \leq b \Longleftrightarrow a \wedge b' = 0 \Longleftrightarrow a' \vee b = 1.$$

In particular,

1. *(Order-reversing)*

$$a \leq b \Longleftrightarrow b' \leq a'.$$

2. *(Atoms) if* $a \in \mathcal{A}(L)$*, then for any* $x \in L$*,*

$$a \leq x \text{ or } a \leq x' \text{ but not both .}$$

Proof :

(i) $\begin{cases} 1 \vee 0 = 1 \\ 1 \wedge 0 = 0 \end{cases} \Longrightarrow 0' = 1, 1' = 0$ and $\begin{cases} a' \wedge a'' = 0 \\ a' \vee a'' = 1 \end{cases} \Longrightarrow a'' = a$

(ii)
- $(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = (0 \wedge b) \vee (a \wedge 0) = 0 \vee 0 = 0$
  $(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = (b' \vee 1) \wedge (a' \vee 1) = 1 \wedge 1 = 1.$
  Then $(a \wedge b)' = a' \vee b'$.
- $(a \vee b) \vee (a' \wedge b') = (a \vee b \vee a') \wedge (a \vee b \vee b') = (0 \vee b) \wedge (a \vee 0) = 0 \wedge 0 = 0$
  $(a \vee b) \wedge (a' \wedge b') = (a \wedge a' \wedge b') \vee (b \wedge a' \wedge b') = (b' \wedge 1) \vee (a' \wedge 1) = 1 \vee 1 = 1.$
  Then $(a \vee b)' = a' \wedge b'$.

(iii)
- $a \leq b \Longleftrightarrow a \wedge b' = 0$
  $(\Longrightarrow)$ $a \wedge b' \leq b \wedge b' = 0$, then $a \wedge b' = 0$.
  $(\Longleftarrow)$ If $a \wedge b' = 0$, then we have:

$$\begin{aligned} b \vee (a \wedge b') = b \vee 0 = b &\Longrightarrow (b \vee b') \wedge (a \vee b) = b \\ &\Longrightarrow 1 \wedge (a \vee b) = b \\ &\Longrightarrow a \vee b = b \\ &\Longrightarrow a \leq b. \end{aligned}$$

- $a \leq b \Longleftrightarrow a' \vee b = 1.$
  $(\Longrightarrow)$ $1 = a \vee a' \leq a' \vee b$, then $a' \vee b = 1$.
  $(\Longleftarrow)$ If $a' \vee b = 1$, then we have:

$$\begin{aligned} a \wedge (a' \vee b) = a \wedge 1 = a &\Longrightarrow (a \wedge a') \vee (a \wedge b) = a \\ &\Longrightarrow 0 \vee (a \wedge b) = a \\ &\Longrightarrow a \wedge b = a \\ &\Longrightarrow a \leq b. \end{aligned}$$

1. (Order-reversing)

$$\begin{aligned} a \leq b &\Longleftrightarrow a \wedge b = a \\ &\Longleftrightarrow (a \wedge b)' = a' \\ &\Longleftrightarrow a' \vee b' = a' \\ &\Longleftrightarrow b' \leq a'. \end{aligned}$$

2. (Atoms)

Let $a \in \mathcal{A}(L)$ and for any $x \in L$, suppose $a \leq x$ and $a \leq x'$. Then $a \leq x \wedge x'$, then $a \leq 0$, then contradiction.

Chapter 2

Residuated lattices

In this chapter, we study certain necessary concepts: residuated mappings, closures and Galois connection. In particular, we discuss residuated lattice and a list of their basic properties.

2.1 Residuated mappings

Residuated mappings play a central role throughout this exposition, with fundamental concepts being presented whenever possible in terms of natural properties of them.

Definition 2.1 (**Residuated mappings**). *[10]*
Let E and F be two posets. A map $f\colon E \longrightarrow F$ is residuated if there exists a map $g\colon F \longrightarrow E$ such that the following holds for any $x \in F$ and any $y \in F$:

$$f(x) \leq y \Longleftrightarrow x \leq g(y).$$

Proposition 2.1. *[19]*
Let E and F be two sets, if $f\colon E \longrightarrow F$ is a residuated mapping, then an isotone mapping g: $F \longrightarrow E$ which is such that $g \circ f \geq id_{\mathcal{E}}$ and $f \circ g \leq id_{\mathcal{F}}$ is in fact unique.

Proof :
Suppose g and h are each isotone and satisfy these properties.
Then $g = id_{\mathcal{E}} \circ g \leq (h \circ f) \circ g = h \circ (f \circ g) \leq h \circ id_{\mathcal{F}} = h$. So $g \leq h$.
Similarly, $h \leq g$ and therefore $g = h$.

Remark 2.1.

- *f and g form a resituated pair, and that g is a residual of f.*
- *We shall denote this unique g by f^+.*

Example 2.1. *[19]*

1. *If E is any set and $A \subseteq E$, then $\lambda_{\mathcal{A}}$: $\mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ defined by $\lambda_{\mathcal{A}}(X) = A \cap X$ is residuated with residual $\lambda^+_{\mathcal{A}}$ given by $\lambda^+_{\mathcal{A}}(Y) = Y \cup A$.*
2. *For $m \in \mathbb{N}^*$ define f_m: $\mathbb{N} \longrightarrow \mathbb{N}$ by $f_m(n) = mn$. Then f_m is residuated with $f^+_m(p) = \lfloor p/m \rfloor$ where $\lfloor q \rfloor$ denote the integer part of $q \in \mathbb{Q}$.*

Lemma 2.1. *[10]*
If f: $E \longrightarrow F$ and f^+: $F \longrightarrow E$ form a residuated pair, then

(1) $f^+(y) = max\{x \in E : f(x) \leq y\}$.

(2) $f(x) = min\{y \in F : x \leq f^+(y)\}$.

(3) $f \circ f^+ \circ f = f$ and $f^+ \circ f \circ f^+ = f^+$.

Proof :

(1) Obviously, $f^+(y)$ is an upper bound of $\{x \in E : f(x) \leq y\}$, since $f(x) \leq y$ implies $x \leq f^+(y)$, and it is an element of the set, because $f(f^+(y)) \leq y$.
(2) The proof is obtained dually.
(3) The first equality:
• $f \circ f^+$ is contracting i.e. $(f \circ f^+)(x) \leq x$, by the monotonicity of f then $(f \circ f^+ \circ f)(x) \leq f(x)$, then $f \circ f^+ \circ f \leq f$.
• $f^+ \circ f$ is extensive (i.e. $(f^+ \circ f)(x) \geq x$) and f is monotone, then $f \circ f^+ \circ f \geq f$.

Then $\begin{cases} f \circ f^+ \circ f \leq f \\ f \circ f^+ \circ f \geq f \end{cases} \Longrightarrow f \circ f^+ \circ f = f.$

The second equality is obtained dually.

Lemma 2.2. *[10]*
Suppose that f: $E \longrightarrow F$ and f^+: $F \longrightarrow E$ form a residuated pair. Then f preserves existing joins and f^+ preserves existing meets. Thus, in particular when both E and F are lattices, $f(x \vee x\prime) = f(x) \vee f(x\prime)$ for all $x, x\prime \in E$ and $f^+(y \wedge y\prime) = f^+(y) \wedge f^+(y\prime)$ for all $y, y\prime \in F$.

Proof :

Assume that the join $\vee X$ of a subset X of E exists. We will show that $f(\vee X)$ is the least upper bound of the set $f[X]$. By the monotonicity of f, $f(\vee X)$ is an upper bound of $f[X]$. If y is an upper bound of $f[X]$, then $f(x) \leq y$ for all $x \in X$. By the residuation property $x \leq f^{+}(y)$, for all $x \in X$, so $\vee X \leq f^{+}(y)$. Again by residuation, we have $f(\vee X) \leq y$.
Likewise, f^{+} preserves existing meets.

Lemma 2.3. *[10]*
Suppose that both E and F are complete lattices and that f is a map from E to F. Then, f is residuated if and only if, f preserves all (possibly infinite) joins. Dually, a map f^{+} from to F to E is the residual of a map f if and only if, f^{+} preserves all (possibly infinite) meets.

Proof :

One direction follows from **Lemma** 2.2. For the converse, assume that f preserves all *joins*. Define $f^{+}(y) = \vee\{x \in E : f(x) \leq y\}$. Then $f(x) \leq y$ implies $x \leq f^{+}(y)$. Note that f preserves the order, since if $x \leq y$, then $f(y) = f(x \vee y) = f(x) \vee f(y) \geq f(x)$. So, if $x \leq f^{+}(y)$, then $f(x) \leq f(f^{+}(y))$, hence $f(x) \leq f(\vee\{x \in E : f(x) \leq y\}) = \vee\{f(x) \in E : f(x) \leq y\} = y$. Likewise, we prove the statement for f^{+}.

2.2 Closures

We now consider an important type of isotone mapping that is intimately related to a residuated mapping.

Definition 2.2 (**Closure operator**)**.** *[9]*
A closure operator on a lattice L is a map f: $L \longrightarrow L$, that satisfies the following conditions:

(1) f is extensive: $x \leq f(x)$, for all $x \in L$;

(2) f is monotone: if $x \leq y$, then $f(x) \leq f(y)$, for all $x, y \in L$;

(3) f is idempotent: $f(f(x)) = f(x)$, for all $x \in L$.

Example 2.2.

Let $(X, \sigma(X))$ *a topological space and* $\sigma(X) = \{set\ of\ overt\}$. $\overline{X}$ *is a closure operator because:*

(1) *For all* $X \in \sigma(X)$, *we have* $X \subseteq \overline{X}$.

(2) *For all* $A, B \in \sigma(X)$, *if* $A \subseteq B$, *then* $\overline{A} \subseteq \overline{B}$.

(3) *For all* $X \in \sigma(X)$, *we have* $\overline{\overline{X}} = \overline{X}$.

Definition 2.3 (Interior operator). *[9]*

An interior operator on a lattice L is a map f: $L \longrightarrow L$, *that satisfies the following conditions:*

(1) f *is contracting:* $f(x) \leq x$, *for all* $x \in L$;

(2) f *is monotone: if* $x \leq y$, *then* $f(x) \leq f(y)$, *for all* $x, y \in L$;

(3) f *is idempotent:* $f(f(x)) = f(x)$, *for all* $x \in L$.

Example 2.3.

Let $(X, \sigma(X))$ *a topological space and* $\sigma(X) = \{set\ of\ overt\}$. $Int(X)$ *is a interior operator because:*

(1) *For all* $X \in \sigma(X)$, *we have* $Int(X) \subseteq X$.

(2) *For all* $A, B \in \sigma(X)$, *if* $A \subseteq B$, *then* $Int(A) \subseteq Int(B)$.

(3) *For all* $X \in \sigma(X)$, *we have* $Int(Int(X)) = Int(X)$.

Lemma 2.4. *[10]*

If f: $E \longrightarrow F$ *and* f^+*:* $F \longrightarrow E$ *form a residuated pair, then* $f^+ \circ f$ *is a closure operator and* $f \circ f^+$ *is an interior operator.*

Proof :

For $x \in E$, $f(x) \leq f(x)$, so $x \leq f^+(f(x))$ by the residuation property. Likewise, $f \circ f^+(x) \leq x$. If $x, y \in E$ and $x \leq y$, then $x \leq y \leq f^+(f(y))$, so $f(x) \leq f(y)$ by residuation. Likewise, f^+ is monotone. Then $f^+ \circ f$ and $f \circ f^+$ are monotone. Moreover, for all $x \in E$, $f \circ f^+(f(x)) \leq f(x)$, since $f \circ f^+$ is contracting; so $f^+(f(f^+(f(x)))) \leq f^+(f(x))$, since f^+ is monotone. The reverse inequality follows from the fact that $f^+ \circ f$ is extensive, hence $f^+ \circ f$ is idempotent. Likewise, $f \circ f^+$ is idempotent.

2.3 Galois connections

We now present a concept that is intimately related to that of a residuated mapping.

Definition 2.4 (**Galois connection**). *[19]*
Let E, F two ordered sets and an antitone mappings f: $E \longrightarrow F$ and g: $F \longrightarrow E$, we say that the pair (f, g) establishes a Galois connection between E and F if $f \circ g \geq id_{\mathcal{F}}$ and $g \circ f \geq id_{\mathcal{E}}$.

Example 2.4. *[19]*
Let S be a semi-group with a zero element 0. For every $A \subset S$ define respectively the left and right annihilators of A by:

$$L(A) = \{x \in S : (\forall a \in A)\ xa = 0\} \text{ and } R(A) = \{x \in S : (\forall a \in A)\ ax = 0\}.$$

If $A = \{x\}$ we shall write $L(x)$ for $L(\{x\})$, and similarly $R(x)$ for $R(\{x\})$. The mappings $L, R : \mathcal{P}(\mathcal{S}) \longrightarrow \mathcal{P}(\mathcal{S})$ so defined are both antitone and as can readily be verified, we have $id_{\mathcal{P}(\mathcal{S})} \leq L \circ R$ and $id_{\mathcal{P}(\mathcal{S})} \leq R \circ L$. Then (L, R) is a Galois connection.

2.4 Residuated lattice

A residuated lattice is an algebraic structure which consists both lattice and monoid structures, and has binary operations called residuations. In this case, we consider the mappings f_y, f_y^+: $L \longrightarrow L$ defined by $f_y(x) = x * y$ and $f_y^+(x) = x \rightarrow y$, for any $x, y \in L$.

2.4.1 Basic concepts

We review some basic concepts are needed in the later sections.

Definition 2.5 (**Residuated lattice**). *[16]*
A residuated lattice is an algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$, or simply, $(L, *, \rightarrow)$ where

(RL1) $(L, \wedge, \vee, 0, 1)$ *is a lattice (the corresponding order will be denoted by $\leq$) with the least element 0 and the greatest element 1;*

(RL2) $(*, \rightarrow)$ *forms an adjoint couple on L, i.e. for any $a, b, c \in L$:*

- *If $a \leq b$ and $c \leq d$ then $a * c \leq b * d$,*

- *If* $b \leq c$ *then* $a \to b \leq a \to c$,
- *If* $a \leq b$ *then* $b \to c \leq a \to c$,
- $a * b \leq c \Longleftrightarrow a \leq b \to c$ *(adjointness condition).*

(RL3) $(L, *, 1)$ *forms a commutative monoid, i.e. for any* $a, b, c \in L$,

- $(a * b) * c = a * (b * c)$;
- $a * b = b * a$;
- $1 * a = a$.

Example 2.5.

Let $L = \{0, a, b, c, 1\}$ *with* $0 < a, b < c < 1$, *but* a *and* b *are incomparable.*

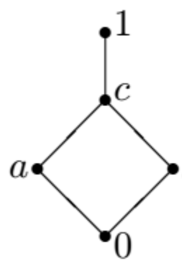


Figure 2.1

Then $(L, \wedge, \vee, *, \to, 0, 1)$ *is a residuated lattice, where* $\to$ *and* $*$ *are defined as in the tables:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Proposition 2.2.

Let $(L, \wedge, \vee, *, \to, 0, 1)$ *be a residuated lattice, for all* $x, y \in L$:

(i) $x \to y = max\{z \in L : x * z \leq y\}$.

(ii) $x * y = min\{z \in L : x \leq y \to z\}$.

Proof :

Let $(L, \wedge, \vee, *, \to, 0, 1)$ is a residuated lattice, then we have (RL1), (RL2) and (RL3) .

(i) We have : $x \to y \leq x \to y$, then by (RL2) $x*(x \to y) \leq y$, then $x \to y \in \{z \in L : x*z \leq y\}$.

Let $m \in \{z \in L : x * z \leq y\}$, then $x * m \leq y \Longleftrightarrow m \leq x \to y$ (by(RL2)).

Finally, $x \to y = max\{z \in L : x * z \leq y\}$.

(ii) we have $y * x \leq y * x$, then by(RL2) $x \leq y \to (y * x)$, and $y * x = x * y$ ($*$ is commutative), then $x * y \in \{z \in L : x \leq y \to z\}$.

Let $t \in \{z \in L : x \leq y \to z\}$, then $x \leq y \to t \Longleftrightarrow y * x \leq t \Longleftrightarrow x * y \leq t$ (by (RL2)).

Finally $x * y = min\{z \in L : x \leq y \to z\}$.

Definition 2.6. *[6]*

A totally ordered (linearly ordered) residuated lattice will be called chain.

Definition 2.7. *[2]*

- *A residuated lattice is called distributive if it has a distributive lattice.*

- *A residuated lattice is called complete if L is a lattice complete.*

- *A residuated lattice satisfies the prelinearity axiom if and only if,* $(x \to y) \vee (y \to x) = 1$ *holds.*

- *A residuated lattice is divisible if and only if,* $x \wedge y = x * (x \to y)$*. It can be shown that divisibility is equivalent to the following condition: for each* $x \leq y$ *there is* z *such that* $x = y * z$.

- *A residuated lattice satisfies the law of double negation (and is called integral) if and only if,* $x = (x \to 0) \to 0$ *holds.*

- *A residuated lattice has square roots if there is an unary operation* $^{1/2}$ *satisfying:*
  (1) $x^{1/2} * x^{1/2} = x$;
  (2) $y * y \leq x$ *implies* $y \leq x^{1/2}$.

- *A Heyting algebra is a residuated lattice where* $x * y = x \wedge y$.

- *A BL-algebra is a residuated lattice which is divisible and satisfies the prelinearity axiom.*

- *An MV-algebra is a residuated lattice in which* $x \vee y = (x \to y) \to y$ *holds.*

2.4.2 Some properties of residuated lattices

We always suppose that L is a bounded lattice with the smallest element 0 and greatest element 1, $*$ and $\to$ are two binary operations on L. In addition, we often use the following derived operations:

$$a^0 = 1, a^n = a^{n-1} * a, n \in \mathbb{N}^*, a, b \in L.$$

Proposition 2.3.

Let L be a residuated lattice, then we have the following proprieties for all $x, y, z \in L$:

(R1) $x \leq y \Longleftrightarrow x \to y = 1$;

(R2) $x \to 1 = 1$;

(R3) $x \to x = 1$;

(R4) $x \leq y \to x$;

(R5) $1 \to x = x$;

(R6) $\begin{cases} x \to y = 1 \\ y \to x = 1 \end{cases} \Longleftrightarrow x = y$;

(R7) $x * (x \to y) \leq y$, $x \leq (x \to y) \to y$;

(R8) $x * y \leq x, y$;

(R9) $x \leq y$ *implies,* $y \to z \leq x \to z$, $z \to x \leq z \to y$;

(R10) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$;

(R11) $x * (y \to z) \leq y \to (x * z) \leq (x * y) \to (x * z)$.

Proof :

(R1) ($\Longrightarrow$) It suffices to see that $x * 1 = x \leq y$, this implies $1 \leq x \to y$ (by (RL2)), then $x \to y = 1$.

($\Longleftarrow$)

$$\begin{aligned} x \to y = 1 &\Longrightarrow \forall t \quad t \leq x \to y \\ &\Longrightarrow \forall t \quad x * t \leq y \qquad (by \ \ (RL2)) \\ &\Longrightarrow t = 1 \quad x * 1 \leq y \\ &\Longrightarrow x \leq y \qquad (because\ (L, *, 1)\ is\ a\ monoid). \end{aligned}$$

(R2) We have $x \leq y \Longleftrightarrow x \to y = 1$, we replace y by 1, we obtain $x \leq 1 \Longleftrightarrow x \to 1 = 1$.

(R3) We have $x \leq x$, then by (R1) $x \to x = 1$.

(R4) We have $y \leq 1$, then $y \leq x \to x$ (because $x \to x = 1$), then by (RL2) $y * x \leq x$, then $x \in \{z : y * z \leq x\}$ with the greatest element is $y \to x$, then $x \leq y \to x$.

(R5) We have $y \to x \leq y \to x$, then $y * (y \to x) \leq x$, we replace y by 1, we obtain $1 * (1 \to x) \leq x$, then $1 \to x \leq x$. And by (R4) we have $x \leq 1 \to x$. So $1 \to x = x$.

(R6) We have $\begin{cases} x \to y = 1 \\ y \to x = 1 \end{cases} \Longleftrightarrow \begin{cases} x \leq y \\ y \leq x \end{cases} \Longleftrightarrow x = y.$

(R7) • We have $x \to y \leq x \to y$, then by (RL2) $x * (x \to y) \leq y$.

• We have $x * (x \to y) \leq y$, then by (RL2) $x \leq (x \to y) \to y$.

(R8) We have $x \leq x$ and $y \leq 1$, then $x * y \leq x$ ($*$ is monotone).

Similarly for $x * y \leq y$.

(R9) • We prove if $x \leq y$ then $y \to z \leq x \to z$.

We have $x \to z = max\{t \in L : y * t \leq z\}$.

$x \to z$ is an upper bound of the set $\{t \in L : y * t \leq z\}$, then $t \leq x \to z$. Let $t \in L$ such that $y * t \leq z$, then by (RL2) $t \leq y \to z$. Then $y \to z \leq x \to z$ ($x \to z$ is an upper bound).

• Using the same method if $x \leq y$ then $z \to x \leq z \to y$.

(R10) • We prove $x \to (y \to z) = y \to (x \to z)$

$t \leq x \to (y \to z)$ iff $x * t \leq y \to z$ iff $(x * t) * y \leq z$ iff $x * (t * y) \leq z$ iff $t * y \leq x \to z$ iff $t \leq y \to (x \to z)$.

we have $\begin{cases} x \to (y \to z) \leq x \to (y \to z) \\ y \to (x \to z) \leq y \to (x \to z) \end{cases} \Longrightarrow \begin{cases} y \to (x \to z) \leq x \to (y \to z) \\ x \to (y \to z) \leq y \to (x \to z) \end{cases} \Longrightarrow x \to (y \to z) = y \to (x \to z)$.

• We prove $x \to (y \to z) = (x * y) \to z$

$t \leq (x * y) \to z$ iff $(x * y) * t \leq z$ iff $x * t \leq y \to z$ iff $t \leq x \to (y \to z)$.

We have $\begin{cases} x \to (y \to z) \leq x \to (y \to z) \\ (x * y) \to z \leq (x * y) \to z \end{cases} \Longrightarrow \begin{cases} x \to (y \to z) \leq (x * y) \to z \\ (x * y) \to z \leq x \to (y \to z) \end{cases} \Longrightarrow x \to (y \to z) = (x * y) \to z$.

Then $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$.

(R11) • We prove $x * (y \to z) \leq y \to (x * z)$:

We have $y \to z \leq y \to z$, then by (RL2) $y * (y \to z) \leq z$ then $(x * y) * (y \to z) \leq x * z$ then $x * (y \to z) \leq y \to (x * z)$.

• We prove $y \to (x * z) \leq (x * y) \to (x * z)$:

We have by (R8) $x * y \leq y$, then by (R9) $y \to (x * z) \leq (x * y) \to (x * z)$.

Proposition 2.4. *[16]*

Let $(L, *, \to)$ a residuated lattice. The following two properties hold:

(i) $x * (y \wedge z) \leq (x * y) \wedge (x * z)$*;*

(ii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

Proof :

(i) We have $\begin{cases} y \wedge z \leq y \\ y \wedge z \leq z \end{cases} \Longrightarrow \begin{cases} x * (y \wedge z) \leq x * y \\ x * (y \wedge z) \leq x * z \end{cases} \Longrightarrow x * (y \wedge z) \leq (x * y) \wedge (x * z)$.

(ii) • Let $F = \{t \in L : (x \to y) \wedge (x \to z) \leq x \to t\}$ with the greatest element denoted by t'.

We have $\begin{cases} y \in F & (\textit{because } (x \to y) \wedge (x \to z) \leq x \to y) \\ z \in F & (\textit{because } (x \to y) \wedge (x \to z) \leq x \to z) \end{cases} \Longrightarrow \begin{cases} y \leq t' \\ z \leq t' \end{cases} \Longrightarrow y \wedge z \leq t' \Longrightarrow y \wedge z \in F \Longrightarrow (x \to y) \wedge (x \to z) \leq x \to (y \wedge z)$.

• We have $\begin{cases} y \wedge z \leq y \\ y \wedge z \leq z \end{cases} \overset{by\ (R9)}{\Longrightarrow} \begin{cases} x \to (y \wedge z) \leq x \to y \\ x \to (y \wedge z) \leq x \to z \end{cases} \Longrightarrow x \to (y \wedge z) \leq (x \to y) \wedge (x \to$

$z)$.

Then $\begin{cases} x \to (y \wedge z) \leq (x \to y) \wedge (x \to z) \\ (x \to y) \wedge (x \to z) \leq x \to (y \wedge z) \end{cases} \Longrightarrow x \to (y \wedge z) = (x \to y) \wedge (x \to z).$

Proposition 2.5. *[6]*

In a residuated lattice L the following assertions are equivalent :

(i) $x^2 = x$ *for every* $x \in L$*;*

(ii) $x * (x \to y) = x * y = x \wedge y$ *for every* $x, y \in L$.

Proof :

$(i) \Longrightarrow (ii)$ Let $x, y \in L$, we have by (R11)

$$\begin{aligned} x * (x \to y) \leq (x * x) \to (x * y) &\Longleftrightarrow x * (x \to y) \leq x \to (x * y) \\ &\Longleftrightarrow x \to y \leq x \to (x \to (x * y)) \quad (byRL2) \\ &\Longleftrightarrow x \to y \leq (x * x) \to (x * y) \quad (by\ R10) \\ &\Longleftrightarrow x \to y \leq x^2 \to (x * y) \\ &\Longleftrightarrow x \to y \leq x \to (x * y) \\ &\Longleftrightarrow x * (x \to y) \leq x * y. \end{aligned}$$

We have by (R4) $y \leq x \to y$, then $x * y \leq x * (x \to y)$, so $x * (x \to y) = x * y$.

To prove $x * y = x \wedge y$:

We have by (R8) $x * y \leq x, y$, then $x * y \leq x \wedge y$.

We have $x \wedge y \leq x, y$, then $x \wedge y = (x \wedge y)^2 \leq x * y$.

Then $x * y = x \wedge y$.

$(ii) \Longrightarrow (i)$ It suffices to take $x = y$ we obtain $x * x = x \wedge x = x \Longleftrightarrow x^2 = x$.

Proposition 2.6. *[2]*

$(L, \wedge, \vee, *, \to, 0, 1)$ *is a residuated lattice if and only if,*

(i) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice;*

(ii) $(L, \to, 1)$ *satisfies:*

$$x = 1 \to x,$$

$$x \to (y \to z) = y \to (x \to z).$$

(iii) $*$ *and* $\rightarrow$ *satisfy the adjointness property.*

Proof :

$(\Longrightarrow)$ It is easy to see that (i), (ii), and (iii) hold in any residuated lattice.

$(\Longleftarrow)$ It suffices to show that (i), (ii), and (iii) imply that $(L, *, 1)$ is a commutative monoid. Let $x, y, z, t \in L$, we have:

- $x * 1 \leq t$ iff $x \leq 1 \rightarrow t$ iff $x \leq t$ (because $t = 1 \rightarrow t$).
  Then we have $\begin{cases} x * 1 \leq x * 1 \\ x \leq x \end{cases} \Longrightarrow \begin{cases} x * 1 \leq x \\ x \leq x * 1 \end{cases} \Longrightarrow x * 1 = x$.
- $x * y \leq t$ iff $x \leq y \rightarrow t$ iff $1 \leq x \rightarrow (y \rightarrow t)$ iff $1 \leq y \rightarrow (x \rightarrow t)$ (by (ii)) iff $y \leq x \rightarrow t$ iff $y * x \leq t$.
  Then we have $\begin{cases} x * y \leq x * y \\ y * x \leq y * x \end{cases} \Longrightarrow \begin{cases} x * y \leq y * x \\ y * x \leq x * y \end{cases} \Longrightarrow x * y = y * x$.
- $x * (y * z) \leq t$ iff $y * z \leq x \rightarrow t$ iff $y \leq z \rightarrow (x \rightarrow t)$ iff $y \leq x \rightarrow (z \rightarrow t)$ (by (ii)) iff $x * y \leq z \rightarrow t$ iff $(x * y) * z \leq t$.
  Then we have $\begin{cases} x * (y * z) \leq x * (y * z) \\ (x * y) * z \leq (x * y) * z \end{cases} \Longrightarrow \begin{cases} x * (y * z) \leq (x * y) * z \\ (x * y) * z \leq x * (y * z) \end{cases} \Longrightarrow x * (y * z) = (x * y) * z$.

Then $(L, *, 1)$ is a commutative monoid.

Theorem 2.1. *[4]*

Let $W_k = (0, 1, ..., k-1)$*. Then* $(W_k, \vee, \wedge, *, \rightarrow, 0, k-1)$*, where*

$x \wedge y = min\{x, y\}$, $\quad x * y = \begin{cases} 0 & \textit{if } x + y \leq k-1; \\ (x+y) - (k-1) & \textit{if } x + y > k-1. \end{cases}$

$x \vee y = max\{x, y\}$, $\quad x \rightarrow y = \begin{cases} k-1 & \textit{if } x \leq y; \\ (k-1) - x + y & \textit{if } x > y. \end{cases}$

is a residuated lattice.

Proof :

- It is easy to see that $(W_k, \vee, \wedge, *, \rightarrow, 0, k-1)$ is a bounded lattice.
- Since, $+$ is commutative and associative, then $*$ is commutative and associative. Also, for any $x \neq 0$, $x + (k-1) > k-1$ implies that $x * (k-1) = x + (k-1) - (k-1) = x$, i.e. $(k-1)$

is the identity and so $(W_k, *, (k-1))$ is a commutative monoid.

• Hence, it is enough to show that $(*, \rightarrow)$ are adjoint pair. Let $x, y, z \in W_k$ and $x * y \leq z$.

If $y \leq z$, then $x \leq k - 1 = y \rightarrow z$.

If $y > z$:

- If $x * y = 0$, then $x + y \leq k - 1$, then $x \leq (k-1) - y \leq (k-1) - y + z = y \rightarrow z$.

- If $x*y = (x+y)-(k-1)$, then $x*y = (x+y)-(k-1) \leq z$, then $x \leq (k-1)-y+z = y \rightarrow z$.

Conversely, let $x \leq y \rightarrow z$.

If $x + y \leq k - 1$, then $x * y = 0 \leq z$

If $x + y > k - 1$:

- If $y \leq z$, then $x + y \leq x + z$, and so we get $k - 1 < x + y \leq x + z \leq (k-1) + z$, thus $x + y \leq (k-1) + z$, then $x * y = x + y - (k-1) \leq z$

- If $y > z$, then $x \leq y \rightarrow z = (k-1) - y + z$, then $x * y = x + y - (k-1) \leq z$.

2.4.3 Morphism of residuated lattices

Definition 2.8 (**Morphism**). *[15]*

Let L_1 and L_2 be two residuated lattices. A function h: $L_1 \longrightarrow L_2$ is called a morphism of residuated lattices if and only if, it is a morphism of bounded lattices and a morphism of monoids and it satisfies: for all $x, y \in L_1$, $h(x \rightarrow y) = h(x) \rightarrow h(y)$.

Definition 2.9 (**Isomorphism**).

An isomorphism between two residuated lattices is a morphism of residuated lattices bijective.

Theorem 2.2. *[6]*

All residuated lattices with the same cardinality are isomorphic.

Proof :

Let $(L_1, \wedge_{L_1}, \vee_{L_1}, *_{L_1}, \rightarrow_{L_1}, 0_{L_1}, 1_{L_1})$ be a residuated lattice and let L_2 be the set with the same cardinality as L_1. Then there exists a bijection f: $L_1 \longrightarrow L_2$, put $0_{L_2} = f(0_{L_1})$ and $1_{L_2} = f(1_{L_1})$. Since for any $y_1, y_2 \in L_2$ there exist $x_1, x_2 \in L_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ on L_2 we can define the following hyper-operations:

$$y_1 \vee_{L_2} y_2 = f(x_1 \vee_{L_1} x_2), y_1 \wedge_{L_2} y_2 = f(x_1 \wedge_{L_1} x).$$

$$y_1 *_{L_2} y_2 = f(x_1 *_{L_1} X_2), y_1 \rightarrow_{L_2} y_2 = f(X_1 \rightarrow_{L_1} X_2).$$

$(L_2, \wedge_{L_2}, \vee_{L_2}, *_{L_2}, \rightarrow_{L_2}, 0_{L_2}, 1_{L_2})$ is a residuated lattice. The map φ: $L_1 \longrightarrow L_2$ defined by $\varphi(x) = f(x)$ is an isomorphism between these two residuated lattices.

2.4.4 Filters and ideals of residuated lattices

Definition 2.10 (Filter). *[14]*

A non-empty subset F of a residuated lattice L is called a filter if it satisfies, for any x, y in L

(F1) $(x \in F, y \in F) \Longrightarrow x * y \in F$;

(F2) $(x \leq y,\ x \in F) \Longrightarrow y \in F$.

Example 2.6.

1. For any residuated lattice L, $\{1\}$ and L are filters of L.

*2. Let $L = \{0, a, b, c, 1\}$ a residuated lattice, the Hasse diagram of L is defined in **Figure** 2.1. Then $F_1 = \{c, 1\}$ and $F_2 = \{a, c, 1\}$ are filters of L.*

Definition 2.11. *[11]*

Residuated lattice L is called simple, if the only filters of L are $\{1\}$ and L.

Proposition 2.7. *[5]*

Let L be a residuated lattice, $G \subseteq L$ a non-empty subset and $a \in L$, then

(i) F_G is a filter generated by G such that, $F_G = \{x \in L\colon a_1 * ... * a_n \leq x,\ \forall a_1, ..., a_n \in G\}$.

(ii) F_a is called principal filter generated by a such that, $F_a = \{x \in L\colon a^n \leq x, \forall n \in \mathbb{N}^*\}$.

Proof :

(i) • $G \subset F_G$:

Let $a_1 \in G$, we have $a_1 \in L$ and $a_1^n = \underbrace{a_1 * ... * a_1}_{n \text{ terms}} \leq a_1$, then $a_1 \in F_G$. Then $G \subset F_G$.

• F_G is a filter:

Let $x, y \in F_G$, then there exists $a_i, b_j \in G$ $(i = 1, ..., n; j = 1, ..., m)$ such that $a_1 * ... * a_n \leq x$ and $b_1 * ... * b_m \leq y$, then $a_1 * ... * a_n * b_1 * ... * b_m \leq x * y$ ($*$ is monotone), then $x * y \in F_G$. Then we have (F1).

Let $x, y \in L$ and $x \leq y$. Let $x \in F_G$, then $a_1 * ... * a_n \leq x$, $\forall a_1, ..., a_n \in G$, then $a_1 * ... * a_n \leq y$, then $y \in F_G$. Then we have (F2).

- F_G is the smallest filter of L containing G:

  Let F a filter of L containing G. Let $x \in F_G$, then $a_1 * ... * a_n \leq x$ for all $a_1, ..., a_n \in G$, then $a_1 * ... * a_n \leq x$ for all $a_1, ..., a_n \in F$ (because $G \subset F$), then $a_1 * ... * a_n \in F$ (because F is a filter), then $x \in F$. Then $F_G \subset F$.

(ii) It suffices to take $G = \{a\}$, we obtain $F_G = F_a$.

Definition 2.12.

Let L be a residuated lattice.

- *We say that $F \subseteq L$ is a maximal filter if*

1. *$F \neq L$;*
2. *For any filter $X \subseteq L$, $F \subseteq X \subseteq L \Longrightarrow X = F$ or $X = L$.*

- *We say that $F \subseteq L$ is a prime filter if*

1. *$F \neq L$;*
2. *If $x, y \in L$ and $x \vee y \in F$, then $x \in F$ or $y \in F$.*

Definition 2.13. *[16]*

A residuated lattice is said to be local if and only if, it has exactly one maximal filter.

Notation 2.1.

Looking ahead, we shall adopt the following neater notation:

- *$x' = x \rightarrow 0$.*
- *$x \oplus y = x' \rightarrow y$.*

Definition 2.14 (Ideal). *[14]*

Let L be a residuated lattice and $I \subseteq L$. I is said to be an ideal of L if I satisfies:

(I1) For any $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$;

(I2) For any $x, y \in I$, $x \oplus y \in I$.

Example 2.7.

1. For any residuated lattice L, $\{0\}$ and L are ideals of L.

*2. Let $L = \{0, a, b, c, 1\}$ a residuated lattice, the Hasse diagram of L is defined in **Figure** 2.1. Then $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are ideals of L.*

Theorem 2.3. *[14]*

Let L be a residuated lattice. I is an ideal of L if and only if, I satisfies following conditions:

(I3) $0 \in I$;

(I4) For any $x, y \in L$, if $x' * y \in I$ and $x \in I$, then $y \in I$.

Proof :

Suppose I is an ideal of L. It follows from (I1) that $0 \in I$, so (I3) holds. Let $x, y \in L$ such that $x' \oplus y \in I$ and $x \in I$. Observe that $y \to (x \oplus (x' * y)) = y \to (x' \to (x' * y)) = (y * x') \to (y * x') = 1$ (by (R10)), we have $y \leq x \oplus (x' * y)$. As $x' * y \in I$ and $x \in I$, it follows from (I2) that $x \oplus (x' * y) \in I$, hence by (I1) $y \in I$. Therefore, (I4) holds.

Conversely, Let $x, y \in L$ such that $x \leq y$ and $y \in I$, then $y' \leq x'$ and $y' * x \leq x' * x = (x \to 0) * x = x * (x \to 0) \leq 0$ (by (R7)), then $x' * x$, so $y' * x = 0 \in I$, by (I4) $y \in I$ we have $x \in I$, that is (I1) holds.

Assume $x, y \in I$. Since $x' * (x \oplus y) = x' * (x' \to y) \leq y$ (by (R7)) and $y \in I$, by (I1) we have $x' * (x \oplus y) \in I$. It follows from (I4) that $x \oplus y \in I$.

Theorem 2.4. *[14]*

Let G be a non-empty set of a residuated lattice L. Then

$$I_G = \{a \in L : a \leq (...((x_1 \oplus x_2) \oplus x_3)...) \oplus x_n, x_i \in G, i = 1, 2, ..., n\}.$$

Proof :

• $G \subset I_G$:

Let $a_1 \in G$, we have by (R8) $a_1 * (...((a_1 \oplus a_1) \oplus a_1)...) \oplus a_1)' \leq a_1$, then $a_1 \leq (...((a_1 \oplus a_1) \oplus a_1)...) \oplus a_1)' \to a_1$, hence $a_1 \leq (...((a_1 \oplus a_1) \oplus a_1)...) \oplus a_1) \oplus a_1$, then $a_1 \in I_G$. So $G \subset F_G$.

• I_G is an ideal:

Obviously, $0 \in I_G$. Let $x' * y \in I_G$ and $x \in I_G$, then there exists $a_i, b_j \in G$ $(i = 1, ..., n; j = 1, ..., m)$ such that $x' * y \leq (...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n$ and $x \leq (...((b_1 \oplus b_2) \oplus b_3)...) \oplus b_m$.

We have $x' \geq ((...((b_1 \oplus b_2) \oplus b_3)...) \oplus b_m)'$

and

$$\begin{aligned} y &\leq x' \to (...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n \\ &\leq ((...((b_1 \oplus b_2) \oplus b_3)...) \oplus b_m)' \to (...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n \\ &\leq (...((b_1 \oplus b_2) \oplus b_3)...) \oplus b_m \oplus (...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n. \end{aligned}$$

Therefore $y \in I_G$ and so I_G is an ideal of L.

• I_G is the smallest ideal of L containing G:

Let I an ideal of L and $G \subset I$. For any $x \in I_G$ there exists $a_i \in G$ $(i = 1, ..., n)$ such that $x \leq (...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n$. Since for all $i = 1, ..., n$ we have $a_i \in I$, then by (I2) $(...((a_1 \oplus a_2) \oplus a_3)...) \oplus a_n \in I$, then by (I1) $x \in I$. Then $I_G \subset I$.

Corollary 2.1. *[14]*

For any element a of a residuated lattice L, we have

$$I_a = \{a \in L : a \leq (\dots \underbrace{((a \oplus a) \oplus a)...) \oplus a}_{n\ terms}, n \in \mathbb{N}^*\}.$$

Chapter 3

Heyting algebras

In this chapter, we study some important concepts: implicative algebra, positive implicative algebra. In the last, we study Heyting algebra and some proprieties.

3.1 Implicative algebra

Definition 3.1 (Implicative algebra).
An algebra $(A, \to, 1)$ of type $(2, 0)$ called implicative algebra if it satisfies the following conditions:

(I1) $a \to a = 1$*;*

(I2) $\begin{cases} a \to b = 1 \\ b \to c = 1 \end{cases} \Longrightarrow a \to c = 1$*;*

(I3) $\begin{cases} a \to b = 1 \\ b \to a = 1 \end{cases} \Longrightarrow a = b$*;*

(I4) $a \to 1 = 1$*.*

Example 3.1.

- *Every chain with the greatest element is an implicative algebra.*

Proposition 3.1.

We define $a \leq b \Longleftrightarrow a \to b = 1$.

$(A, \leq)$ *is an ordered set with the greatest element* 1 *if and only if,* A *is an implicative algebra.*

Proof :

Let A an ordered set, then we have:

- $a \to a = 1$ ($a \leq a$);
- $\begin{cases} a \to b = 1 \\ b \to c = 1 \end{cases} \Longrightarrow \begin{cases} a \leq b \\ b \leq c \end{cases} \Longrightarrow a \leq c \Longrightarrow a \to c = 1;$
- $\begin{cases} a \to b = 1 \\ b \to a = 1 \end{cases} \Longrightarrow \begin{cases} a \leq b \\ b \leq a \end{cases} \Longrightarrow a = b;$
- $a \to 1 = 1$ (because $\forall a \in A$ $a \leq 1$).

Conversely, if we consider an implicative algebra A, then

- Reflexivite: $a \leq a$ ($a \to a = 1$);
- Antisymetric: $\begin{cases} a \leq b \\ b \leq a \end{cases} \Longrightarrow \begin{cases} a \to b = 1 \\ b \to a = 1 \end{cases} \Longrightarrow a = b;$
- Transitivité: $\begin{cases} a \leq b \\ b \leq c \end{cases} \Longrightarrow \begin{cases} a \to b = 1 \\ b \to c = 1 \end{cases} \Longrightarrow a \to c = 1 \Longrightarrow a \leq c;$
- $\forall a \in A, a \leq 1$ ($a \to 1 = 1$).

Proposition 3.2 (Modus ponens).

$$\begin{cases} a = 1 \\ a \to b = 1 \end{cases} \Longrightarrow b = 1.$$

Proof :

$$\begin{cases} b \to 1 = 1 & (by I4) \\ 1 \to b = 1 \end{cases} \overset{(I3)}{\Longrightarrow} b = 1.$$

Remark 3.1.

The rule of modus ponens is valid for all logic.

3.2 Positive implicative algebra

Definition 3.2 (Positive implicative algebra).

An algebra $(A, \rightarrow, 1)$ *of type* $(2,0)$ *called positive implicative algebra if it satisfies the following conditions:*

(P1) $a \rightarrow (b \rightarrow a) = 1$;

(P2) $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$;

(P3) $\begin{cases} a \rightarrow b = 1 \\ b \rightarrow a = 1 \end{cases} \Longrightarrow a = b$;

(P4) $a \rightarrow 1 = 1$.

Example 3.2.

Let $(X, \sigma(X))$ *a topological space and* $\sigma(x) = \{set\ of\ overt\}$. *Then* $(\sigma(X), \rightarrow, 1)$ *is a positive implicative algebra such that for all* $A, B \in \sigma(X)$, $A \rightarrow B = Int(B \cup \mathcal{C}A)$

Proposition 3.3.

Any positive implicative algebra is an implicative algebra.

Proof :

(I1): Replace c by a in the (P2) we obtained:

$(a \rightarrow (b \rightarrow a)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow a)) = 1$

$1 \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow a)) = 1$ ($a \rightarrow (b \rightarrow a) = 1$ by (P1))

$(a \rightarrow b) \rightarrow (a \rightarrow a) = 1$ (by modus ponens)

Replace b by $a \rightarrow a$

$(a \rightarrow (a \rightarrow a)) \rightarrow (a \rightarrow a) = 1$

$1 \rightarrow (a \rightarrow a) = 1$ (because $a \rightarrow (a \rightarrow a) = 1$ by (P1))

$a \rightarrow a = 1$ (by modus ponens).

(I2): Suppose $a \rightarrow b = 1$ and $b \rightarrow c = 1$

$(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$ (by (P2))

$(a \rightarrow 1) \rightarrow (1 \rightarrow (a \rightarrow c)) = 1$

$1 \rightarrow (1 \rightarrow (a \rightarrow c)) = 1$

$1 \rightarrow (a \rightarrow c) = 1$ (by modus ponens)

$a \to c = 1$ (by modus ponens).

We have (I3) by (P3) and (I4) by (P4).

Remark 3.2.

The inverse of the ***Proposition*** *3.3 is false.*

Example 3.3.

Let $A = \{0, a, b, 1\}$, *where* $\to$ *defined in the table:*

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	0	1	1
1	0	0	0	1

Then $(A, \to, 1)$ *is an implicative algebra but is not a positive implicative algebra because* $a \to (b \to a) = 0 \neq 1$.

Proposition 3.4.

In a positive implicative algebra the following conditions hold for all $a, b, c \in A$:

(P5) $a \leq b \to a$;

(P6) $a \leq b \to c \Longleftrightarrow b \leq a \to c$;

(P7) $a \leq (a \to b) \to b$;

(P8) $1 \to a = a$;

(P9) if $b \leq c$ *then* $a \to b \leq a \to c$;

(P10) if $a \leq b$ *then* $b \to c \leq a \to c$;

(P11) $a \to (a \to b) = a \to b$;

(P12) $a \to (b \to c) = b \to (a \to c)$;

(P13) $(b \to c) \leq (a \to b) \to (a \to c)$;

(P14) $a \to b \le (b \to c) \to (a \to c)$.

Proof :

(P5) by (P1) $a \to (b \to a) = 1$, then $a \le b \to a$.

(P6) ($\Longrightarrow$) Suppose $a \le b \to c$, then $a \to (b \to c) = 1$, and we have $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$ (P2). Then by modus ponens: $(a \to b) \to (a \to c) = 1$.

$$\begin{cases} b \le a \to b & (P1) \\ (a \to b) \le (a \to c) \end{cases} \Longrightarrow b \le a \to c.$$

($\Longleftarrow$) Suppose $b \le a \to c$, then $b \to (a \to c) = 1$, hence $(b \to a) \to (b \to c) = 1$ (by modus ponens and (P2))

$$\begin{cases} a \le b \to a & (P1) \\ b \to a \le b \to c \end{cases} \Longrightarrow a \le b \to c.$$

(P7) We have $a \to b \le a \to b$, then $a \le (a \to b) \to b$ (by (P6)).

(P8) by (P7) $a \le (a \to b) \to b$ we replace a by 1 and b by a, we obtain $1 \le (1 \to a) \to a$, hence $(1 \to a) \to a = 1$, so $1 \to a \le a$. And by $(P5)$ we have $a \le 1 \to a$.

Then $\begin{cases} 1 \to a \le a \\ a \le 1 \to a \end{cases} \Longrightarrow 1 \to a = a.$

(P9) If $b \le c$, then $b \to c = 1$. in this case $a \to (b \to c) = a \to 1 = 1$ or $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$ (by (P2)), then by modus ponens $(a \to b) \to (a \to c) = 1$. Then if $b \le c$, hence $a \to b \le a \to c$.

(P10) If $a \le b$, then $a \to b = 1$ and by (P2) we have $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$, hence $(a \to (b \to c)) \to (1 \to (a \to c)) = 1$ but by (P8) $1 \to (a \to c) = (a \to c)$. Then $(a \to (b \to c)) \to (a \to c) = 1$, so $(a \to (b \to c)) \le (a \to c)$

$$\begin{cases} b \to c \le a \to (b \to c) & (by (P5)) \\ a \to (b \to c) \le a \to c \end{cases} \Longrightarrow b \to c \le a \to c.$$

Then if $a \le b$, hence $b \to c \le a \to c$.

(P11) We have $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$ (P2) replace c by b, we obtain

$(a \to (a \to b)) \to ((a \to a) \to (a \to b)) = 1$ and then

$(a \to (a \to b)) \to (1 \to (a \to b)) = 1$ ($a \to a = 1$)

$(a \to (a \to b)) \to (a \to b) = 1$ ($1 \to (a \to b) = (a \to b)$ by (P8)), then $a \to (a \to b) \leq a \to b$.

By (P1) we have $(a \to b) \to (a \to (a \to b)) = 1$, then $a \to b \leq a \to (a \to b)$.

Then $\begin{cases} a \to (a \to b) \leq a \to b \\ a \to b \leq a \to (a \to b) \end{cases} \Longrightarrow a \to (a \to b) = a \to b$.

(P12) By (P2) we have $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,

then $a \to (b \to c) \leq (a \to b) \to (a \to c) \qquad (*)$

• $\begin{cases} b \leq a \to b & (by(P1)) \\ a \to b \leq (a \to (b \to c)) \to (a \to c) & (by(P6)) \end{cases} \Longrightarrow b \leq (a \to (b \to c)) \to (a \to c) \overset{(P6)}{\Longrightarrow} a \to (b \to c) \leq b \to (a \to c)$.

• In $(*)$ we replace a by b and b by a, we obtain $b \to (a \to c) \leq (b \to a) \to (b \to c)$.

$\begin{cases} b \to a \leq (b \to (a \to c)) \to (b \to c) & (by(P6)) \\ a \leq b \to a & (by(P1)) \end{cases} \Longrightarrow a \leq (b \to (a \to c)) \to (b \to c) \overset{(P6)}{\Longrightarrow} b \to (a \to c) \leq a \to (b \to c)$.

Then $\begin{cases} a \to (b \to c) \leq b \to (a \to c) \\ b \to (a \to c) \leq a \to (b \to c) \end{cases} \Longrightarrow a \to (b \to c) = b \to (a \to c)$.

(P13) We have $\begin{cases} a \to (b \to c) \leq (a \to b) \to (a \to c) & (by(P2)) \\ b \to c \leq a \to (b \to c) & (by(P1)) \end{cases} \Longrightarrow b \to c \leq (a \to b) \to (a \to c)$.

(P14) By (P13): $(b \to c) \leq (a \to b) \to (a \to c)$ and (P6) we have then $a \to b \leq (b \to c) \to (a \to c)$.

3.3 Heyting algebra

Definition 3.3 (Heyting algebra).

A Heyting algebra is a lattice $(E, \leq, \wedge, \vee)$ satisfying: (p) $\forall a, b \in E$, the set: $\{x \in E : a \wedge x \leq b\}$ has a greatest element that will be denoted $a \to b$ pseudo complement of a with respect to b.

$a \wedge (a \to b) \leq b$

$a \wedge x \leq b \Longleftrightarrow x \leq a \to b$

Example 3.4.

1. *Let X be a bounded chain with $|X| \geq 3$. We define a binary operation $\to$ on X by:*
   $x \to y = \begin{cases} 1 & if\ x \leq y \\ y & otherwise. \end{cases}$
   Then X is a Heyting algebra.

2. *Every Boolean algebra is a Heyting algebra, with $x \to y$ given by $x' \vee y$.*

Definition 3.4.

A Heyting algebra A is said to be commutative if it satisfies $(a \to b) \to b = (b \to a) \to a$ for every $a, b \in A$.

Proposition 3.5.

Every Heyting algebra is a distributive lattice.

Proof :

Let A a Heyting algebra, we prove $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ for all $a, b, c \in A$.

We have $\begin{cases} b \leq b \vee c \\ c \leq b \vee c \end{cases} \Longrightarrow \begin{cases} a \wedge b \leq a \wedge (b \vee c) \\ a \wedge c \leq a \wedge (b \vee c) \end{cases} \Longrightarrow (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$.

And we have $\begin{cases} a \wedge b \leq (a \wedge b) \vee (a \wedge c) \\ a \wedge c \leq (a \wedge b) \vee (a \wedge c) \end{cases} \Longrightarrow \begin{cases} b \leq a \to (a \wedge b) \vee (a \wedge c) \\ c \leq a \to (a \wedge b) \vee (a \wedge c) \end{cases} \Longrightarrow b \vee c \leq a \to (a \wedge b) \vee (a \wedge c) \Longrightarrow a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

Then $\begin{cases} (a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c) \\ a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c) \end{cases} \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Remark 3.3.

- *A Heyting algebra has a greatest element.*
- *A Heyting algebra has not necessarily a 0.*

Proposition 3.6.

Every Heyting algebra is a positive implicative algebra.

Proof :

(P1): Let $F = \{x : b \wedge x \leq a\}$, the greatest element of F is $b \to a$.

We have $b \wedge a \leq a$, then $a \in F$, hence $a \leq b \to a$. Consequently $a \to (b \to a) = 1$.

(P2): We take $x = a \to b$, $y = a \to c$ and $z = b \to c$.

$$\begin{cases} a \wedge x = a \wedge (a \to b) \leq b \\ a \wedge (a \to z) \leq z \end{cases} \Longrightarrow a \wedge x \wedge (a \to z) \leq b \wedge z = b \wedge (b \to c) \leq c$$

$\Longrightarrow a \wedge x \wedge (a \to z) \leq c$

$\Longrightarrow x \wedge (a \to z) \leq a \to c$

$\Longrightarrow x \wedge (a \to z) \leq y$

$\Longrightarrow a \to z \leq x \to y$

$\Longrightarrow a \to (b \to c) \leq (a \to b) \to (a \to c)$

$\Longrightarrow a \to (b \to c) \to (a \to b) \to (a \to c) = 1.$

(P3): $\begin{cases} a \to b = 1 \\ b \to a = 1 \end{cases} \Longrightarrow \begin{cases} a \leq b \\ b \leq a \end{cases} \Longrightarrow a = b.$

(P4) We have $a \leq b \Longleftrightarrow a \to b = 1$, then we replace b by 1, we obtain $a \leq 1 \Longleftrightarrow a \to 1 = 1$.

Proposition 3.7.

A Heyting algebra has the following properties:

(H1) $a \to a = 1.$

(H2) $(a \to c) \wedge (b \to c) \leq (a \vee b) \to c.$

(H3) $a \wedge (a \to b) = a \wedge b.$

(H4) $b \wedge (a \to b) = b.$

(H5) $a \to (b \wedge c) = (a \to b) \wedge (a \to c).$

Proof :

(H1) It suffices to see that $a \wedge 1 = a \leq a$, this implies $1 \leq a \to a$, then $a \to a = 1$.

(H2) $a \wedge (a \to c) \wedge (b \to c) \leq a \wedge (a \to c) \leq c$ (because $a \to c \leq a \to c$)

$b \wedge (a \to c) \wedge (b \to c) \leq b \wedge (b \to c) \leq c$ (because $b \to c \leq b \to c$)

$\Longrightarrow (a \vee b) \wedge (a \to c) \wedge (b \to c) = (a \wedge (a \to c) \wedge (b \to c)) \vee (b \wedge (a \to c) \wedge (b \to c)) \leq c \vee c = c.$

Then we have $(a \vee b) \wedge (a \to c) \wedge (b \to c) \leq c \Longleftrightarrow (a \to c) \wedge (b \to c) \leq (a \vee b) \to c$.

(H3) We have $\begin{cases} b \leq a \to b \\ a \leq a \end{cases} \Longrightarrow a \wedge b \leq a \wedge (a \to b) \qquad (*).$

And $\begin{cases} a \wedge (a \to b) \leq b \\ a \wedge (a \to b) \leq a \end{cases} \Longrightarrow a \wedge (a \to b) \leq a \wedge b \qquad (**).$

Then $(*)$ and $(**)$ give $a \wedge (a \to b) = a \wedge b$.

(H4) $\begin{cases} b \leq a \to b \\ b \leq b \end{cases} \Longrightarrow b \leq b \wedge (a \to b)$

$a \to b \leq 1 \Longrightarrow a \to b \leq b \to b$ (because $b \to b = 1$)$\Longrightarrow b \wedge (a \to b) \leq b$.

Then $\begin{cases} b \leq b \wedge (a \to b) \\ b \wedge (a \to b) \leq b \end{cases} \Longrightarrow b \wedge (a \to b) = b.$

(H5)

- $a \to (b \wedge c) \leq a \to (b \wedge c) \Longleftrightarrow a \wedge (a \to (b \wedge c)) \leq b \wedge c$ $\begin{cases} a \wedge (a \to (b \wedge c)) \leq c \\ a \wedge (a \to (b \to c)) \leq b \end{cases} \Longleftrightarrow \begin{cases} a \to (b \wedge c) \leq a \to c \\ a \to (b \wedge c) \leq a \to b \end{cases} \Longrightarrow a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$.
- $\begin{cases} (a \to b) \wedge (a \to c) \leq a \to b \\ (a \to b) \wedge (a \to c) \leq a \to c \end{cases} \Longleftrightarrow \begin{cases} a \wedge ((a \to b) \wedge (a \to c)) \leq b \\ a \wedge ((a \to b) \wedge (a \to c)) \leq c \end{cases} \Longleftrightarrow a \wedge ((a \to b) \wedge (a \to c)) \leq b \wedge c \Longleftrightarrow (a \to b) \wedge (a \to c) \leq a \to (b \wedge c)$.

  Then $\begin{cases} a \to (b \wedge c) \leq (a \to b) \wedge (a \to c) \\ (a \to b) \wedge (a \to c) \leq a \to (b \wedge c) \end{cases} \Longrightarrow a \to (b \wedge c) = (a \to b) \wedge (a \to c).$

Conclusion

In this work, we have discussed some concepts of the ordered sets theory, lattices and Boolean algebra.

And then we tried to study residuated lattice and we prove some of its properties.

We finished this work by Heyting algebra.

Bibliography

[1] A. Belhadj, *Génération de Treillis et propriétés algébriques* Mémoire de Magistère, Université : Mouloud Mammeri, Tizi-Ouzou, 03/11/2011.

[2] R. Belohlavek, *Some properties of residuated lattices*, Czechoslovak Math. j. 53 (123) (2003), 161-171.

[3] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, 2005.

[4] R. A. Borzooei, *Fundamental residuated Lattices*, Quasigroups and Related Systems 22 (2014), 179-192.

[5] H. L. Constantin,*Some properties of filters in Stonean residuated lattices*, IOSR Journal of Engineering (IOSRJEN), Vol. 04, Issuse 02, (2014), 44-60.

[6] R. Cretan and A. Jeflea, *On the lattice of congruence filters of a residuated lattice*, Annals of University of Craiova, Math. Sci. Ser. 33 (2006), 174-188.

[7] B. A. Davey and H. A. Periestley, *Introduction to Lattices and order*, Second edition, Cambridge University Press, Cambridge, 2002.

[8] L. Frécon, *Éléments de Mathématiques discrètes*, Presses polytechniques et universitaires romandes, CH-1015 Lausanne, 2002.

[9] N. Galatos, *Varieties of residuated lattices*, Dissertation, Submitted to the Faculty of the Graduate School of Vanderbilt University in partial fulfillment of the requirements of the degree of Doctor of Philosophy, May, 2003.

[10] N. Galatos, P. Jipsen, T.Kowalski and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, First edition, Elsivier, 2007.

[11] S. Ghorbani and L. Torkzadeh, *Nilpotent Elements of Residuated Lattices*, International Journal of Mathematics and Mathematical science, Vol. 2012, Article ID 763428, 9 pages, (2012).doi: 10. 1155/2012/763428.

[12] B. Kolman, R. C. Busby and S. C. Ross, *Discrete Mathematical Structures*, Fourth edition, Pearson education, 2001.

[13] S. Lipschutz, *Discrete Mathematics*, Third edition, Mcgra-Whill, 2007.

[14] Y. Liu, Y. Qin, X. Qin and Y. Xu, *Ideals and fuzzy ideals on residuated lattices*, Int. J. Mach. Learn. 8 (2017), 239-253.

[15] C. Muresan, *The Reticulation of a Residuated Lattice*, Bull. Math. Soc. Sci. Math. Roumanie 51 (99) (1) (2008), 47-65.

[16] D. Pei, *Fuzzy Logic Algebras on Residuated Lattices*, Southeast Asian Bulletin of Mathematics 28 (2004), 519–531.

[17] D. Ponasse et J. C. Carrega, *Algèbre et Toboologie boléennes*, Masson, Paris, 1979.

[18] S. Roman, *Lattices and Ordred Sets*, Springer, 2008.

[19] M. Ward, R. P. Dilworth, Residuated lattices, Transactions of the American Mathematical Society, Vol. 45 (1939), 335-354.